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A Proof of a Conjecture of Stanley Concerning Partitions of a Set

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Let n be a positive integer, let Π_n denote the lattice of partitions of $\{1, 2, \ldots, n\}$ and let S_n denote the symmetric group on n letters. For each $S \subseteq \{1, 2, \ldots, n-2\}$ define an S-chain of Π_n to be a 0–1 chain of Π_n having an element of rank s for each $s \in S$ and having no other elements except 0 and 1. The group S_n acts as a group of automorphisms of Π_n hence acts on the set of all S-chains in Π_n. Let $\alpha(S)$ denote the number of orbits of S-chains on Π_n and let $\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|}\alpha(T)$.

It is known that $\beta(S) \geqslant 0$ for all subsets $S \subseteq \{1, 2, \ldots, n-2\}$. Richard Stanley showed that $\beta(\{1, 2, \ldots, n-2\}) = 0$ and that $\sum_S \beta(S)$ is the number of alternating permutations in S_{n-1}. He conjectured that $\beta(S) = 0$ whenever S has the form $\{1, 2, \ldots, j\}$. The aim of this paper is to prove that Stanley's conjecture is true.

1. INTRODUCTION

If k is a positive integer we let $\underline{k}$ denote the set $\{1, 2, \ldots, k\}$. Let S_k denote the symmetric group on k letters and let Π_k denote the lattice of partitions of $\underline{k}$. The group S_k acts on Π_k as follows; if $\sigma \in S_k$ and $B_1/\cdots/B_r \in \Pi_k$ then

$$(B_1/\cdots/B_r)\sigma = (B_1\sigma)/\cdots/(B_r\sigma),$$

where $B_i\sigma = \{b\sigma : b \in B_i\}$. It is easy to check that each $\sigma \in S_k$ is an automorphism of Π_k.

Let P be a ranked poset with 0 and 1. Let $m+1$ be the rank of P. For any set $S \subseteq \underline{m}$ let P_S be the subposet of P consisting of 0, 1 and all elements of P whose rank lies in the set S. An *S-chain* in P is a maximal 0–1 chain of P_S.

Let P be a ranked poset with 0 and 1. Define the *order complex* $\Delta(P)$ to be the abstract simplicial complex whose vertices are the elements of $P-\{0, 1\}$ and whose k-simplices are the chains $x_0 < x_1 < \cdots < x_k$ in $P-\{0, 1\}$. Let $\hat{H}_i(P)$ denote the reduced simplicial homology group $\hat{H}_i(\Delta(P), C)$.

Let G be a group of automorphisms of P where P is a ranked poset of rank $m+1$ with 0 and 1. Then G acts as a group of automorphisms of each P_S (for $S \subseteq \underline{m}$) hence G permutes the S-chains of P. For each $S \subseteq \underline{m}$ let α_S denote the permutation representation of G on the S-chains of P. Let $A_S(P, G)$ be the number of orbits of S-chains of P. So $A_S(P, G)$ is the multiplicity of the trivial representation in α_S. Define the virtual character β_S by

$$\beta = \sum_{T \subseteq S} (-1)^{|S-T|}\alpha_T.$$

The action of G on the S-chains of P gives an action G on the ith reduced homology group of P_S. Let $\gamma_{S,i}: G \to \mathrm{Hom}\,(\hat{H}_i(P_S), \hat{H}_i(P_S))$ be this representation of G. It is known that

$$\beta_S = \sum_i (-1)^{|S|-1-i}\gamma_{S,i}.$$

In the case that P is Cohen–Macaulary, we have $\tilde{H}_i(P_S) = 0$ for $i \neq |S|-1$. Hence if P is Cohen–Macaulay, β_S is equivalent to the natural action of G on the reduced homology group $\tilde{H}_{|S|-1}(P_S)$.

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DEFINITION 1.1. Let P be a Cohen–Macaulay poset with rank $m+1$ and let G be a group of automorphisms of P. Define $\beta(P_S, G)$ to be the multiplicity of the trivial character in the action of G on the unique nonvanishing reduced homology of P_S, $\tilde{H}_{|S|-1}(P_S)$. Equivalently,

$$\beta(P_S, G) = \sum_{T \subseteq S} (-1)^{|S-T|} A_T(P_S, G)$$

EXAMPLE 1.1. Let $P = \Pi_5$ and let $G = S_5$. The rank of P is 4. Table 1 gives $A_S(P, G)$ for all $S \subseteq \{1, 2, 3\}$. It is easy to check that $\beta\ (P_{\underset{\sim}{3}}, G) = 0$

TABLE 1

$\underset{\sim}{S}$	$A_S(P, G)$
$\varnothing$	1
$\{1\}$	1
$\{2\}$	2
$\{3\}$	2
$\{1, 2\}$	2
$\{1, 3\}$	3
$\{2, 3\}$	4
$\{1, 2, 3\}$	5

2. STANLEY'S CONJECTURE

Throughout this section n is a fixed positive integer greater than or equal to 3. For $1 \leqslant j \leqslant n-2$ we let L_j denote $(\Pi_n)_j$. So $L_{n-2} = \Pi_n$. As Π_n is geometric, hence Cohen–Macaulay, each poset L_j has a unique nonvanishing reduced homology. Hence each β_S is an actual character. Stanley made the following conjecture.

CONJECTURE. *The trivial character does not occur in β_j for $1 \leqslant j \leqslant n-2$.*

Stanley proved a special case of the Conjecture which we now discuss in some detail.

Let U be a group, V a subgroup of U and η: $V \to \mathbb{C}$ a character of V. We let $(\eta)_V^U$ denote the representation of U obtained by inducing η to U. Stanley proved the following result about $\beta_{\underline{n-2}}$.

THEOREM 2.1 (*Stanley* [5], *p.* 29). *Let $j = n-2$ so $L_j = \Pi_n$. Let $\sigma = (1, 2, \ldots, n) \in S_n$ and let V be the subgroup of S_n generated by σ. Define the linear character $\eta : V \to \mathbb{C}$* by

$$\eta(\sigma^a) = (e^{2\pi i/n})^a.$$

Let sgn denote the sign representation of S_n. Then

$$\beta_{\underline{n-2}} = \operatorname{sgn}\,(\eta)_v^{S_n}.$$

As η is a 1-dimensional character which is not the sign character ($n \geqslant 3$) we have the next result by Frobenius reciprocity (see Curtis and Reiner [2, p. 271]).

COROLLARY 2.1 (*Stanley*). *The trivial character does not occur in $\beta_{\underline{n-2}}$.*

Notice that Corollary 2.1 is a special case of the Conjecture.

For each $i = 1, 2, \ldots, r$ let U_i be a group and let ρ_i be a representation of U_i on a vector space W_i. Define the representation $\rho_1 \times \cdots \times \rho_r$ of $U_1 \times \cdots \times U_r$ on

$W_1 \times \cdots \times W_r$ as follows. For $(u_1, \ldots, u_r) \in U_1 \times \cdots \times U_r$ and $w_1 \times \cdots \times w_r \in W_1 \times \cdots \times W_r$ define

$$((\rho_1 \times \cdots \times \rho_r)(u_1, \ldots, u_r))(w_1 \times \cdots \times w_r) = (\rho_1(u_1)(w_1)) \times \cdots \times (\rho_r(u_r)(w_r)).$$

Let m denote the multiplicity of the trivial character in $\rho_1 \times \cdots \times \rho_r$ and let m_i denote the multiplicity of the trivial character in ρ_i. It is easy to see that $m = m_1 m_2 \ldots m_r$ hence if the trivial character does not occur in any ρ_i it does not occur in $\rho_1 \times \cdots \times \rho_r$.

Let $n_1, \ldots, n_r$ be positive integers and let L be the geometric lattice $\Pi_{n_1} \times \cdots \times \Pi_{n_r}$. Then L has one nonvanishing reduced homology group over $\mathbb{C}$ which we denote by H. Let G_1 be the group $S_{n_1} \times \cdots \times S_{n_r}$.

By Künneth's formulas (see Spanier [4, p. 228]) we have that

$$H \cong \tilde{H}_{n_1-2}(\Pi_{n_1}, \mathbb{C},) \times \cdots \times \tilde{H}_{n_r-2}(\Pi_{n_r}, \mathbb{C})$$

and that the representation of G_1 on H is the tensor product representation. Thus the next result follows from Corollary 2.1.

COROLLARY 2.2. *The trivial representation of G_1 does not occur as a constituent of G_1 acting on H.*

3. THE PROOF OF STANLEY'S CONJECTURE

In this section we prove Stanley's Conjecture which was given in the preceding section.

THEOREM 3.1. *Let j be an integer with $1 \leq j \leq n-2$. Let $\tilde{H}(j) = \tilde{H}_{j-1}(L_j, \mathbb{Q})$ be the unique nontrivial homology vector space of L_j. Then the trivial representation of S_n does not occur as a constituent of S_n acting on $\tilde{H}(j)$.*

PROOF. It suffices to show that $\beta(L_j, S_n) = 0$. We prove this by induction on j. Consider first the case $j = 1$. It is clear that $A_\emptyset(L_1, S_n) = 1$ and $A_{\{1\}}(L_1, S_n)$ counts the number of orbits of $\hat{0} - \hat{1}$ chains in the rank-2 lattice L_1. The chains are in one-to-one correspondence with the atoms of Π_n. As S_n acts transitively on the atoms of Π_n we have $A_{\{1\}}(L_1, S_n) = 1$. So

$$\beta(L_1, S_n) = -A_\emptyset(L_1, S_n) + A_{\{1\}}(L_1, S_n) = 0.$$

Thus the result holds for $j = 1$.

Assume that j is bigger than 1 and that the result holds for $j-1$. Let T denote the set $T = \{1, 2, \ldots, j-1\}$. By definition,

$$\beta(L_j, S_n) = \sum_{S \subseteq \{1, 2, \ldots, j\}} A_S(L_j, S_n)(-1)^{j+1-|S|}. \tag{3.1}$$

Reorganize the sum (3.1) according to whether or not S is contained in T. Doing so we have

$$\beta(L_j, S_n) = \sum_{S \subseteq T} (-1)^{j+1-|S|} A_S(L_j, S_n) + \sum_{S \subseteq T} (-1)^{j-|S|} A_{S \cup \{j\}}(L_j, S_n).$$

Notice that $\sum_{S \subseteq T} (-1)^{j+1-|S|} A_S(L_j, S_n) = -\beta(L_{j-1}, S_n)$ so by our induction hypothesis

$$\beta(L_j, S_n) = (-1)^j \sum_{S \subseteq T} (-1)^{|S|} A_{S \cup \{j\}}(L_j, S_n). \tag{3.2}$$

Let R denote the jth rank of Π_n. For $\alpha \in R$ let $B(\alpha)$ denote the set of blocks of α and let G_α denote the stabilizer in S_n of α. If $\sigma \in G_\alpha$ then σ induces a permutation of $B(\alpha)$; this permutation is denoted $\hat{\sigma}$. Notice that if $\sigma \in G_\alpha$ and if B_1 and B_2 are blocks of α with $B_1 \hat{\sigma} = B_2$ then $|B_1| = |B_2|$. Let $\Gamma(\alpha)$ denote the subgroup of the symmetric group of $B(\alpha)$ consisting of all $\hat{\sigma}$ with the property that each cycle of $\hat{\sigma}$ contains blocks of the same size. A permutation $\hat{\sigma}$ of $B(\alpha)$ is in $\Gamma(\alpha)$ if and only if $\hat{\sigma}$ is induced on $B(\alpha)$ by some permutation σ in G_α.

Let $S \subseteq T$. Recall that an $S \cup \{j\}$ chain in L_j is a $\hat{0}$–$\hat{1}$ chain in L_j having an element of rank r each $r \in S \cup \{j\}$ and having no other elements except $\hat{0}$ and $\hat{1}$. For $\sigma \in G_\alpha$, $S \subseteq T$ and $\alpha \in R$ let $N(S, \sigma, \alpha)$ denote the number of $S \cup \{j\}$ chains $\hat{0} < \alpha_1 < \cdots < \alpha_{|S|} < \alpha < \hat{1}$ which have α as the unique element of rank j and which satisfy $\alpha_i \sigma = \alpha_i$ for $i = 1, 2, \ldots, |s|$. By Burnside's Counting Lemma we have

$$A_{S\cup\{j\}}(L_j, S_n) = \sum_{\alpha \in R} \frac{1}{n!} \sum_{\sigma \in G_\alpha} N(S, \sigma, \alpha). \tag{3.3}$$

Combining equations (3.2) and (3.3) we have

$$\beta(L_j, S_n) = \frac{1}{n!} \sum_{\alpha \in R} \sum_{\hat{\sigma} \in \Gamma(\alpha)} \sum_{\sigma \in G_\alpha} \sum_{S \subseteq T} (-1)^{j-|S|-1} N(S, \sigma, \alpha), \tag{3.4}$$

where $\sum_{\sigma \in G_\alpha}$ is taken over all $\sigma \in G_\alpha$ which induce the permutation $\hat{\sigma}$ on $B(\alpha)$. To show that $\beta(L_j, S_n) = 0$ we in fact show that for all $\alpha \in R$ and all $\hat{\sigma} \in \Gamma(\alpha)$ we have

$$\sum_{\sigma \in G_\alpha} \sum_{S \subseteq T} (-1)^{j-|S|-1} N(S, \sigma, \alpha) = 0. \tag{3.5}$$

To do so, we first consider the case where $\hat{\sigma}$ has only one cycle in its disjoint cycle decomposition as a permutation of $B(\alpha)$. This case gives us insight into the more difficult general case.

Case 1. $\hat{\sigma} = (B_1, \ldots, B_m)$ where the B_i are blocks of α all of size b and m is the length of the cycle $\hat{\sigma}$.

Suppose $\sigma \in G_\alpha$ and $\sigma|_{B(\alpha)} = \hat{\sigma}$. Then σ maps B_1 to B_2, B_2 to $B_3, \ldots, B_m$ to B_1 and so $\sigma|_{B_i}$ is a 1–1 function from B_i to B_{i+1}. Conversely, given the 1–1 functions:

$$\begin{aligned} &\sigma_1 : B_1 \to B_2 \\ &\sigma_2 : B_2 \to B_3 \\ &\vdots \\ &\sigma_m : B_m \to B_1 \end{aligned} \tag{*}$$

there is a unique permutation $\sigma \in G_\alpha$ such that σ induces the permutation $\hat{\sigma}$ on $B(\alpha)$ and such that $\sigma|_{B_i} = \sigma_i$. Let Z denote the set of m-tuples $(\sigma_1, \ldots, \sigma_m)$ of 1–1 functions satisfying (*); there is a 1–1 correspondence between permutations $\sigma \in G_\alpha$ inducing $\hat{\sigma}$ on $B(\alpha)$ and the elements of Z.

Given $(\sigma_1, \ldots, \sigma_m) \in Z$ the composition $\Pi(\sigma_1, \ldots, \sigma_m) = \sigma_m \circ \sigma_{m-1} \circ \cdots \circ \sigma_1$ is a permutation of B_1. It is easy to check that the map $(\sigma_1, \ldots, \sigma_m) \to \Pi(\sigma_1, \ldots, \sigma_m)$ is a $(b!)^{m-1}$ to 1 correspondence of Z onto the symmetric group of B_1.

Fix $(\sigma_1, \ldots, \sigma_m) \in Z$ and let σ be the permutation of $\{1, 2, \ldots, n\}$ given by this m-tuple. Consider all chains in L_j from $\hat{0}$ to α which are fixed by σ. Such chains are in 1–1 correspondence with the chains from $\hat{0}$ to $\hat{1}$ in Π_{B_1} fixed by $\Pi(\sigma_1, \ldots, \sigma_m)$. The correspondence takes a chain $\hat{0} < \gamma_1 < \cdots < \gamma_t < 1 = B_1$ to the chain $\hat{0} < \gamma_1' < \cdots < \gamma_t' < \alpha$ where the blocks of γ_i' are the m distinct images of the blocks of γ_i under the maps σ_u. If γ_i has s blocks then γ_i' has ms blocks hence

$$\rho(\gamma_i') = m\rho(\gamma_i).$$

It follows that

$$\sum_{\sigma\in G_\alpha}\sum_{S\subseteq T}(-1)^{j-|S|-1}N(S,\sigma,\alpha)$$

$$=\sum_{(\sigma_1,\dots,\sigma_m)\in Z}\ \sum_{S\subseteq\{1,2,\dots,b-2\}}(-1)^{j-|S|-1}N(S,\Pi(\sigma_1,\dots,\sigma_m),1)$$

$$=(-1)^{j-b-1}(b!)^{m-1}\beta(\Pi_b,S_b)=0,$$

the last equality holding by Theorem 2.1. This finishes Case 1.

Case 2. $\hat{\sigma}$ has cycles $M_1,\dots,M_l$. Let m_i denote the length of M_i and let b_i denote the size of the blocks in M_i.

Apply the same argument used in Case 1 to each cycle m_i. Doing so one finds that

$$\sum_{\sigma\in G_\alpha}\sum_{S\subseteq T}(-1)^{j-|S|-1}N(S,\sigma,\alpha)$$

$$=(-1)^{j-1}\left(\prod_{i=1}^{l}(-1)^{b_i}(b_i!)^{m_i-1}\right)\beta((\Pi_{b_1}\times\cdots\times\Pi_{b_l},S_{b_1}\times\cdots\times S_{b_l}))$$

$$=0$$

The last equality holding by Corollary 2.2. This completes the proof of Theorem 3.1.

4. Conclusion

For each set $S\subseteq\{1,2,\dots,n-2\}$ let L_S denote the subset of Π_n consisting of $\hat{0}$, $\hat{1}$ and all elements of Π_n whose rank lies in S. Let $\beta(S)$ denote the integer $\beta(L_S,S_n)$. Stanley has shown that $\beta(S)\geqslant 0$ for all subsets S and that the sum of $\beta(S)$ over all subsets S of $\{1,2,\dots,n-2\}$ is the number of alternating permutations of S_n (an *alternating permutation* σ is one for which the $n-1$ differences $\sigma(2)-\sigma(1),\dots,\sigma(n)-\sigma(n-1)$ have alternating signs). For a proof see Stanley [5], page 28. Thus it is natural to interpret $\beta(S)$ as the number of permutations in S_n having some combinatorial property indexed by the set S. An open problem is to find such a combinatorial interpretation for the numbers $\beta(S)$. Note that the main result of this paper shows that $\beta(S)=0$ whenever S has the form $\{1,2,\dots,j\}$ for $j\leqslant n-2$.

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